Pseudodifferential forms and supermechanics

Denis Kochan
Department of Theoretical Physics,
Faculty of Mathematics Physics and Informatics,
Comenius University,
Mlynská Dolina F2, 842 48 Bratislava, Slovakia
e-mail: kochan@sophia.dtp.fmph.uniba.sk

Abstract

We investigate (pseudo)differential forms in the framework of supergeometry. Definitions, basic properties and Cartan calculus (DeRham differential, Lie derivative, inner product, Hodge operator) are presented; the symplectic supermechanics (even and odd) is formulated; and the question of quantization is discussed. In the framework of supermechanics, we investigate also classical Hamiltonian systems converting to SUSY-QM after quantization.

PACS: 11.30.Pb; 45.10.Na; 02.40.Yy
MSC: 58A10; 58A50
Key words: supergeometry; (pseudo)differential forms; (super)symplectic structures; SUSY-QM

1 Introduction

Supergeometry is surely an interesting and fruitful branch of mathematics with a variety of powerful applications in modern theoretical physics, in particular in SUSY, supergravity and superstrings. From a purely mathematical point of view, supergeometry is a natural extension of the ordinary differential geometry by Grassmann variables. Such anti-commuting extensions represent an essential and inspiring feature of all supermathematics. The first "supermathematician" was undoubtedly Russian mathematician F. A. Berezin, who formulated the main principles of supermathematics (see his famous book [1]). In present days there is a lot of books and articles about supergeometry and its application in physics, but because our aim is not to describe all historical circumstances, we refer only to a few of them [1]-[8] (see also references therein).

Differential forms and Cartan calculus are very effective tools of theoretical physics. It is well known that nine tenths from classical physics may be formulated and investigated from the geometrical point of view. The supergeometrical generalization is therefore very useful when describing systems with both bosonic and fermionic degrees of freedom. The supermathematics (inspired by the physics) is a right tool to do this rigorously. The aim of this paper is to present a very short exposition on pseudodifferential forms (roughly speaking differential forms on an arbitrary supermanifold) and related Cartan calculus. It will be shown that such objects are just functions on another (bigger) supermanifold, therefore they are very easy to handle. All standard geometrical operations will be encoded into distinguished vector fields whereupon the calculus will become simple and beautiful.

2 Pseudodifferential forms

It is well known (for more details see [2]) that the ***pseudodifferential forms*** on arbitrary smooth $m|n$-dimensional supermanifold $\mathfrak{M}$ are defined as functions on $\Pi T\mathfrak{M}$ (odd tangent bundle). The standard differential operations on the forms (DeRham differential, Lie derivative, inner product) are identified with special supervector fields on $\Pi T\mathfrak{M}$. To obtain their exact expressions we use the fruitful idea of Maxim Kontsevich, who pointed out that (see [9])

$$\Pi T\mathfrak{M} \equiv \{ \text{ supermaps: } \mathbb{R}^{0|1} \to \mathfrak{M}\} \ . \tag{1}$$

The supergroup *Diff* $(\mathbb{R}^{0|1})$ defines via its natural right action

$$\Pi T\mathfrak{M} \times \mathit{Diff}\,(\mathbb{R}^{0|1}) \to \Pi T\mathfrak{M} \qquad\qquad (F, g) \mapsto F \circ g$$

the left invariant (fundamental) vector fields $Q,\, E \in \mathfrak{X}(\Pi T\mathfrak{M})$. Their expression in arbitrary local coordinates $(x^i, \xi^\alpha; \psi^i, y^\alpha) = (x^i, \xi^\alpha; \mathrm{d}x^i, \mathrm{d}\xi^\alpha)$ covering the odd tangent bundle ($i = 1, \dots, m$ and $\alpha = 1, \dots, n$) is very simple, namely

$$Q \;=\; \psi^i\partial_{x^i} + y^\alpha\partial_{\xi^\alpha} \quad \textit{DeRham differential} \ , \tag{2}$$

$$E \;=\; \psi^i\partial_{\psi^i} + y^\alpha\partial_{y^\alpha} \quad \textit{Euler field} \ . \tag{3}$$

Throughout the paper, we use left derivatives with respect to Grassmann variables and Einstein's summation convention. The Euler vector field "measures" the degree of homogenity of pseudodifferential forms under the supergroup action, therefore the superalgebra $C^\infty(\Pi T\mathfrak{M})$ has also a $\mathbb{Z}$-graded structure. A direct calculation gives the (super)commutation relations in the superalgebra *diff* $(\mathbb{R}^{0|1})$

$$[Q,\, Q] \;=\; 2Q^2 = 0 \ , \tag{4}$$

$$[E,\, E] \;=\; 0 \ , \tag{5}$$

$$[E,\, Q] \;=\; Q \ . \tag{6}$$

Now we shall very briefly sketch the genesis of Lie derivatives and inner product on pseudodifferential forms under some vector field $V = V^i(x,\xi)\partial_{x^i} + V^\alpha(x,\xi)\partial_{\xi^\alpha} = V(x^i)\partial_{x^i} + V(\xi^\alpha)\partial_{\xi^\alpha} \in \mathfrak{X}(\mathfrak{M})$. The Lie supergroup *Diff* $(\mathfrak{M})$ acts on the supermanifold $\Pi T\mathfrak{M}$,

$$\mathit{Diff}\,(\mathfrak{M}) \times \Pi T\mathfrak{M} \to \Pi T\mathfrak{M} \qquad\qquad (g, F) \mapsto g \circ F \ ,$$

and therefore for arbitrary element V of the superalgebra *diff* $(\mathfrak{M}) = \mathfrak{X}(\mathfrak{M}) = \mathit{Der}\,(C^\infty(\mathfrak{M}))$ we have some vector field $V^\uparrow$ defined on $\Pi T\mathfrak{M}$. Notation $V^\uparrow$ reflects the obvious fact that we have lifted the vector field V from the supermanifold $\mathfrak{M}$ to $\Pi T\mathfrak{M}$. A straightforward coordinate computation[1] gives

$$V^\uparrow = V(x^i)\partial_{x^i} + V(\xi^\alpha)\partial_{\xi^\alpha} + (-1)^{\tilde V} Q(V(x^i))\partial_{\psi^i} + (-1)^{\tilde V} Q(V(\xi^\alpha))\partial_{y^\alpha} \ . \tag{7}$$

The procedure of lifting vector fields preserves parity, $\tilde{V^\uparrow} = \tilde V$.

[1] In the case of an odd vector field ($\tilde V = 1$) it is necessary to consider the *superflow* (homomorphism of the supergroups $\mathbb{R}^{1|1}$ and *Diff* $(\mathfrak{M})$), whose infinitesimal $(\Delta t, \Delta\epsilon)$ version in the coordinates is

$$(x^i;\, \xi^\alpha) \mapsto \Big(x^i + \Delta\epsilon V(x^i) + \frac{\Delta t}{2}[V,\, V](x^i);\ \xi^\alpha + \Delta\epsilon V(\xi^\alpha) + \frac{\Delta t}{2}[V,\, V](\xi^\alpha)\Big) \ .$$

Apart from this natural lifting construction it is also possible to assign to any $V \in \mathfrak{X}(\mathfrak{M})$ certain vector field $V_\uparrow$ on $\Pi T\mathfrak{M}$ such that $\tilde{V_\uparrow} = \tilde{V} + 1$ and

$$[V_\uparrow, Q] = V^\uparrow \ . \tag{8}$$

Obviously, the coordinate expression for $V_\uparrow$ is

$$V_\uparrow = V(x^i)\partial_{\psi^i} + V(\xi^\alpha)\partial_{y^\alpha} \ . \tag{9}$$

It is easy to confirm the validity of supercommutations relations

$$\begin{array}{ll} [E, V^\uparrow] \ = 0 \ , & [E, V_\uparrow] = -V_\uparrow \ , \\ [V^\uparrow, Q] \ = 0 \ , & [V^\uparrow, W^\uparrow] = [V, W]^\uparrow \ , \\ [V_\uparrow, W_\uparrow] = 0 \ , & [V^\uparrow, W_\uparrow] \ = [V, W]_\uparrow \ . \end{array} \tag{10}$$

The vector fields $V^\uparrow$ corresponds to the *Lie derivative* $\mathcal{L}_V$ (with respect to V), whereas $V_\uparrow$ corresponds to the *inner product* i_V (with V). The equation (8) is the famous *Cartan formula*.

An arbitrary pseudodifferential form is a function on the supermanifold $\Pi T\mathfrak{M}$ and therefore it may be expressed in any local coordinates as

$$f = f(x,\xi,\psi,y) = \sum_{\substack{\beta\geq 0\\ k\geq 0}} \sum_{\substack{\alpha_1,\dots,\alpha_\beta\\ i_1,\dots,i_k}} f_{\alpha_1,\dots,\alpha_\beta,i_1,\dots,i_k}(x,y)\xi^{\alpha_1}\dots\xi^{\alpha_\beta}\psi^{i_1}\dots\psi^{i_k} \ , \tag{11}$$

where the ordinary real (complex) valued functions $f_{\alpha_1,\dots,i_k}(x,y)$ are skew-symmetric in the indices $\alpha_1,\dots,i_k$. The Berezin integral (for more details see [1],[2]) of a function f on $\Pi T\mathfrak{M}$,

$$I[f] := \int\limits_{\mathfrak{M}} \overline{\mathrm{d}x\mathrm{d}\xi} \int\limits_{\mathbb{R}^{n|m}} \overline{\mathrm{d}\psi\mathrm{d}y} \ f(x,\xi,\psi,y), \tag{12}$$

defines the *integral* of the pseudodifferential form f over $\mathfrak{M}$. It is clear that such integral is not well defined for all elements of $C^\infty(\Pi T\mathfrak{M})$, because the supermanifold $\mathbb{R}^{n|m}$ (the typical fiber in the bundle $\Pi T\mathfrak{M} \to \mathfrak{M}$) is not compact. The coordinate transformation on the supermanifold $\mathfrak{M}$ $(x^i,\xi^\alpha) \mapsto \big(X^i(x,\xi), \Xi^\alpha(x,\xi)\big)$ induces the corresponding transformation on $\Pi T\mathfrak{M}$,

$$(x^i,\xi^a;\psi^i,y^\alpha) \mapsto \left(X^i,\Xi^\alpha;\Psi^i = \psi^k\frac{\partial X^i}{\partial x^k} + y^\beta\frac{\partial X^i}{\partial \xi^\beta}, Y^\alpha = \psi^k\frac{\partial \Xi^\alpha}{\partial x^k} + y^\beta\frac{\partial \Xi^\alpha}{\partial \xi^\beta}\right) \ .$$

Its berezinian is equal to unity, therefore the integral defined in (12) is coordinate independent.

It is also possible to define (at least formally) Hodge $*$ operator acting on pseudodifferential forms. As in the case of ordinary differential forms, the essential ingredient to define Hodge $*$ operator is metric. The metric may be introduced on an arbitrary $m|n$-dimensional smooth supermanifold $\mathfrak{M}$ (in particular, on an ordinary manifold M) as an even regular (non-degenerate) quadratic function at the tangent bundle $T\mathfrak{M}$

$$G(x,\xi,z,\sigma) = (z^i,\sigma^\alpha)\begin{pmatrix} g_{ij}(x,\xi) & \Gamma_{i\beta}(x,\xi) \\ \Gamma_{\alpha j}(x,\xi) & h_{\alpha\beta}(x,\xi)\end{pmatrix}\begin{pmatrix} z^j \\ \sigma^\beta\end{pmatrix} \ . \tag{13}$$

The non-degeneracy condition reads

$$Ber\ G = det\ (g_{ij} - \Gamma_{i\beta}(h_{\alpha\beta})^{-1}\Gamma_{\alpha j})\, det\ (h_{\alpha\beta})^{-1} \neq 0 \ . \tag{14}$$

The transformation of local coordinates on the supermanifold $\mathfrak{M}$ and the transformation of the coordinates $(x^i, \xi^\alpha, z^i, \sigma^\alpha)$ on $T\mathfrak{M}$ are coupled by

$$(x^i, \xi^a; z^i, \sigma^\alpha) \mapsto \left(X^i, \Xi^\alpha; Z^i = z^k \frac{\partial X^i}{\partial x^k} + \sigma^\beta \frac{\partial X^i}{\partial \xi^\beta}, \Sigma^\alpha = z^k \frac{\partial \Xi^\alpha}{\partial x^k} + \sigma^\beta \frac{\partial \Xi^\alpha}{\partial \xi^\beta} \right) .$$

Let us emphasize that non-degeneracy of G implies that the even skew-symmetric matrix $h_{\alpha\beta}$ is invertible, consequently, $m|n$-dimensional supermanifold $\mathfrak{M}$ may be Riemannian only if n is even (this fact is strongly reminiscent of the situation in symplectic geometry). For the functions on $\Pi T\mathfrak{M}$ (pseudodifferential forms on $\mathfrak{M}$) the *Hodge star operator* is defined via Fourier transformation in the "fibre variables" ψ^i and y^α, namely

$$(*f)(x, \xi, \psi, y) := \int\limits_{\mathbb{R}^{n|m}} \frac{\overline{\mathrm{d}\psi' \mathrm{d}y'}}{\sqrt{Ber\ G}} f(x, \xi, \psi', y')\, e^{-\imath \langle \psi', y' | \psi, y \rangle_G} \ , \tag{15}$$

where the symbol $\langle \psi', y' | \psi, y \rangle_G$ denotes the "scalar product" with respect to the metric G

$$\langle \psi', y' | \psi, y \rangle_G := ({\psi'}^i, {y'}^\alpha) \begin{pmatrix} g_{ij} & \Gamma_{i\beta} \\ \Gamma_{\alpha j} & h_{\alpha\beta} \end{pmatrix} \begin{pmatrix} -\psi^j \\ y^\beta \end{pmatrix} . \tag{16}$$

A straightforward calculation shows that the definition of Hodge $*$ operator is independent on the choice of coordinates, and for ordinary forms it gives a multiple of standard $*_g$. As above, the non-compactness of the fibre $\mathbb{R}^{n|m}$ implies that $*$ is defined only for functions that are behaving well in the variables y^α at infinity, for example for the functions with compact support.

3 Frölicher-Nijenhuis brackets

In this section we will consider only an ordinary (real) smooth m-dimensional manifold M. It has been shown that exterior algebra of differential forms $\Omega(M)$ is in one to one correspondence with $C^\infty(\Pi TM)$. The supergroup *Diff* $(\mathbb{R}^{0|1})$ defines via its action on $C^\infty(\Pi TM)$ also a $\mathbb{Z}$-graded structure on the Lie superalgebra $\mathfrak{X}(\Pi TM) = Der\ (C^\infty(\Pi TM))$, namely

$$V \in \mathfrak{X}^{(k)}(\Pi TM) \Leftrightarrow [E, V] = k\, V =: deg\ (V)\, V \ . \tag{17}$$

The vector fields from $\mathfrak{X}^{(k)}(\Pi TM)$ encode, from the ordinary point of view, the derivations of $\Omega(M)$ of degree k, and moreover, from the (super)Jacobi identity it is clear that $deg\ ([V, W]) = deg\ (V) + deg\ (W)$. The derivations of $\Omega(M)$ commuting with DeRham deferential are represented by special vector fields from $\mathfrak{X}(\Pi TM)$ that commute with Q. The Lie subsuperalgebra of such derivations will be denoted $\mathfrak{X}_Q(\Pi TM)$ (the (super)Jacobi identity states that supercommutator of two Q-invariant vector fields is again Q-invariant). The obvious coordinate expression for $V_A \in \mathfrak{X}_Q^{(k)}(\Pi TM)$ is

$$V_A = A^i(x, \psi) \partial_{x^i} + (-1)^k Q(A^i(x, \psi)) \partial_{\psi^i} \ , \tag{18}$$

where $A^i(x, \psi)$ are functions on ΠTM with the degree of homogenity k (differential forms of kth degree). Using (8) it is possible to assign to any Q-invariant vector field V_A of degree k another vector field $v_A \in \mathfrak{X}^{(k-1)}(\Pi TM)$ (roughly speaking "potential of V_A"), such that

$$v_A = A^i(x, \psi) \partial_{\psi^i} = A^i_{j_1, \ldots, j_k}(x) \psi^{j_1} \ldots \psi^{j_k} \partial_{\psi^i} \ . \tag{19}$$

There is a unique correspondence between $\mathfrak{X}_Q^{(k)}(\Pi TM)$ and total skew-symmetric tensor fields of type $\binom{1}{k}$ on manifold M, because any such tensor A is completely characterized by a set of

component functions $A^i_{j_1,\ldots,j_k}(x)$. We define the *Frölicher-Nijenhuis brackets* between two such tensors A, B of ranks $\binom{1}{k}$, $\binom{1}{l}$ respectively by the supercommutator of corresponding vector fields, namely

$$[V_A,\, V_B] =: V_{\{A,\,B\}_{NB}}\ . \tag{20}$$

All properties of the Frölicher-Nijenhuis brackets may be obtained from the defining equation (20) and from the properties of the supercommutator (for more details see Nijenhuis pioneering work [10]).

4 Symplectic supermechanics

The Cartan calculus is undoubtedly a useful tool in modern theoretical physics. A very nice and simple example of its application is the symplectic formulation of Hamiltonian mechanics (see for example [11],[12]). Our next task is formulation and brief description of the supersymmetric extension (via pseudodifferential forms) of ordinary Hamiltonian mechanics, and quantization of the extended theory. In the general Poisson setting this has been done in [13]-[17].

The supermanifold $\mathfrak{M}$, endowed with a particular function (pseudodifferential form) $\Omega \in C^\infty(\Pi T\mathfrak{M})$, is called

even-symplectic:	*odd-symplectic*:
if $dim(\mathfrak{M}) = 2m\vert n$;	if $dim(\mathfrak{M}) = m\vert m$;
$\tilde{\Omega} = 0$;	$\tilde{\Omega} = 1$;
$Q(\Omega) = 0$ (closedness) ;	$Q(\Omega) = 0$ (closedness) ;
$E(\Omega) = 2\Omega$ (Ω is 2-form) $\Rightarrow$	$E(\Omega) = 2\Omega$ (Ω is 2-form) $\Rightarrow$
Ω is regular polynomial	Ω is regular polynomial
(non-degenerate) of 2-nd	(non-degenerate) of 2-nd
degree in ψ, y .	degree in ψ, y .

The presence of a symplectic structure on the supermanifold $\mathfrak{M}$ allows us to define Poisson brackets on $C^\infty(\mathfrak{M})$: for arbitrary homogeneous function (in the sense of parity) $f \in C^\infty(\mathfrak{M}) \subset C^\infty(\Pi T\mathfrak{M})$, we define the Hamiltonian vector field $\zeta_f \in \mathfrak{X}(\mathfrak{M})$ by the condition

$$\zeta_{f\uparrow}\,\Omega = -(-1)^{\tilde{f}} Q(f)\ . \tag{21}$$

It is evident that $\tilde{\zeta_f} = \tilde{f} + \tilde{\Omega}$ and the assignment $f \mapsto \zeta_f$ is $\mathbb{R}$-linear. The corresponding *Poisson brackets* are

$$\{f,\, g\}_{\tilde{\Omega}} := \zeta_{f\uparrow}\zeta_{g\uparrow}\,\Omega(-1)^{\tilde{\Omega}+\tilde{g}} = \zeta_f^{\uparrow}\, g(-1)^{\tilde{\Omega}+1} = \zeta_f\, g(-1)^{\tilde{\Omega}+1}\ . \tag{22}$$

Following formulas are valid

$$\widetilde{\{f,\, g\}_{\tilde{\Omega}}} = \tilde{f} + \tilde{g} + \tilde{\Omega}\ , \tag{23}$$
$$\zeta_{\{f,\,g\}_{\tilde{\Omega}}} = [\zeta_f,\, \zeta_g]\ , \tag{24}$$
$$\{f,\, g+h\}_{\tilde{\Omega}} = \{f,\, g\}_{\tilde{\Omega}} + \{f,\, h\}_{\tilde{\Omega}}\ , \tag{25}$$
$$\{f,\, g\}_{\tilde{\Omega}} = -\{g,\, f\}_{\tilde{\Omega}}(-1)^{(\tilde{f}+\tilde{\Omega})(\tilde{g}+\tilde{\Omega})}\ , \tag{26}$$
$$\{f,\, gh\}_{\tilde{\Omega}} = \{f,\, g\}_{\tilde{\Omega}}h + g\{f,\, h\}_{\tilde{\Omega}}(-1)^{(\tilde{f}+\tilde{\Omega})\tilde{g}}\ , \tag{27}$$
$$0 = \{f,\, \{g,\, h\}_{\tilde{\Omega}}\}_{\tilde{\Omega}}(-1)^{(\tilde{f}+\tilde{\Omega})(\tilde{h}+\tilde{\Omega})} + \text{cyclic permutations}\ . \tag{28}$$

Observables of supermechanics are by definition functions on $\mathfrak{M}$; $C^\infty(\mathfrak{M})$ is $\mathbb{Z}_2$-graded commutative algebra and at the same time Lie superalgebra, and the compatibility of these two structures is guaranteed by the graded Leibniz rule (27).

To define the dynamics we need the Hamiltonian $H \in C^\infty(\mathfrak{M})$ (homogeneous element in the sense of parity). The (super)time evolution is generated by (super)flow $\Phi_{(t,\epsilon)}(\zeta_H)$ of the corresponding Hamiltonian vector field ζ_H; the general formula for the pull-back of the (super)flow on the observables is

$$f \mapsto f_{(t,\epsilon)} := \begin{cases} e^{-t\zeta_H^\uparrow} f & \text{when } \tilde{H} + \tilde{\Omega} = 0 \ , \\ e^{-\epsilon\zeta_H^\uparrow - \frac{t}{2}\zeta^\uparrow_{\{H,H\}_{\tilde{\Omega}}}} f & \text{when } \tilde{H} + \tilde{\Omega} = 1 \ . \end{cases} \tag{29}$$

It is convenient to rewrite the integral formula for the (super)time[2] evolution (29) into a differential one. This amounts to writing down the Hamiltonian equations of motion

$$\text{if } \tilde{H} + \tilde{\Omega} = 0 \text{ then } \{H,\, f\}_{\tilde{\Omega}}(-1)^{\tilde{\Omega}} = \partial_t f \ , \tag{30}$$

$$\text{if } \tilde{H} + \tilde{\Omega} = 1 \text{ then } \{H,\, f\}_{\tilde{\Omega}}(-1)^{\tilde{\Omega}} = (\partial_\epsilon + \epsilon\partial_t) f \ . \tag{31}$$

The question of symmetry of the Hamiltonian system $(\mathfrak{M},\, \Omega,\, H)$ is also very simple: vector field $V \in \mathfrak{X}(\mathfrak{M})$ is a Cartan symmetry if $V^\uparrow \Omega = 0 = V^\uparrow H$. If moreover there exists some function $F \in C^\infty(\mathfrak{M})$ for which $V_\uparrow \Omega = -(-1)^{\tilde{F}} Q(F)$, then we call V an exact Cartan symmetry and the function F is a conserved quantity.

The quantization with a given symplectic structure may be performed by the famous Fedosov construction [18], well known from deformation quantization theory, which for the simplest case (flat phase space, discussed in more detail below) coincides with the Wigner-Moyal-Weyl quantization (for more detail see [19]).

Quantization of even supersymplectic structures: We shall sketch shortly the quantization procedure for supermanifold $\mathfrak{M} = \mathbb{R}^{2m|n}$ (equipped with global coordinates $(x^1, \ldots, x^m, x^{m+1} = p_1, \ldots, x^{2m} = p_m; \xi^1, \ldots, \xi^n)$), the superanalog of ordinary phase space, endowed with the symplectic structure (pseudodifferential form)

$$\Omega = \psi^i \pi_i - \frac{1}{2} g_{\alpha\beta} y^\alpha y^\beta = \mathrm{d}x^i \wedge \mathrm{d}p_i - \frac{1}{2} g_{\alpha\beta} \mathrm{d}\xi^\alpha \wedge \mathrm{d}\xi^\beta \ , \tag{32}$$

where $g_{\alpha\beta} = \mathrm{diag}(+1, \ldots, +1, -1, \ldots, -1)$. This is a natural extension of the canonical 2-form $\omega = \mathrm{d}x^i \wedge \mathrm{d}p_i$ on $\mathbb{R}^{2m}$. It is not so difficult to prove that the famous Darboux theorem, well known from standard symplectic geometry (see for example [20]), is valid in supersymplectic case, too, so that an arbitrary $2m|n$-dimensional even symplectic supermanifold $\mathfrak{M}$ is locally isomorphic to $\mathbb{R}^{2m|n}$ with the symplectic form (32). Let us stress that even "supersymplecticity" (contrary to "supermetricity") does not lead to any restriction on n. The corresponding coordinate Hamiltonian vector fields

$$\zeta_{x^i} = \partial_{p_i} \ , \quad \zeta_{p_i} = -\partial_{x^i} \ , \quad \zeta_{\xi^\alpha} = -g^{\alpha\beta}\partial_{\xi^\beta} \ , \tag{33}$$

determine elementary Poisson brackets

$$\begin{array}{lll} \{x^i,\, x^j\} = 0 & \{p_i,\, p_j\} = 0 & \{p_i,\, x^j\} = \delta_i^j \ , \\ \{x^i,\, \xi^\alpha\} = 0 & \{p_i,\, \xi^\alpha\} = 0 & \{\xi^a,\, \xi^\beta\} = g^{\alpha\beta} \ . \end{array} \tag{34}$$

The canonical operator quantization is equivalent, via Weyl isomorphism, to quantization with operator symbols (ordinary functions on the phase space) and vice versa, with the operator product

[2] We see that superflow forces us to extend the ordinary evolution of classical mechanics in $t \in \mathbb{R}$ (additive Lie group) to superevolution in $(t, \epsilon) \in \mathbb{R}^{1|1}$ (one of the simplest Lie supergoups). The multiplication in $\mathbb{R}^{1|1}$ is given by $(t, \epsilon) \cdot (t', \epsilon') = (t + t' + \epsilon\epsilon', \epsilon + \epsilon')$, with the neutral element $\mathrm{e} = (0, 0)$ and the inverse element $(t, \epsilon)^{-1} = (-t, -\epsilon)$. Lie superalgebra $\mathbf{r}^{1|1}$ is generated by vector fields $V_0 = \partial_t$ and $V_1 = \partial_\epsilon + \epsilon\partial_t$ obeying $[V_i,\, V_j] = 2ijV_0$.

replaced by the star product[3]

$$\begin{aligned} f\star h &= fh+\frac{\imath\hbar}{2}\{f,\,h\}+o(\hbar) \\ &= fh+\frac{\imath\hbar}{2}\left(\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial x^i}-\frac{\partial f}{\partial x^i}\frac{\partial h}{\partial p_i}+(-1)^{\tilde f+1}\frac{\partial f}{\partial \xi^\alpha}g^{\alpha\beta}\frac{\partial h}{\partial \xi^\beta}\right)+o(\hbar)\ , \end{aligned} \tag{35}$$

and the supercommutator $[\hat f,\,\hat h]_\mp$ is in one to one correspondence with a star (super)bracket $\{f,\,h\}_{\star_\mp} := f\star h \mp h\star f$. The "star product technique" leads to the canonical supercommutation relations

$$\begin{aligned} &[\hat x^i,\,\hat x^j]_- = 0\ , \quad [\hat p_i,\,\hat p_j]_- = 0\ , \quad [\hat p_i,\,\hat x^j\,]_- = \imath\hbar\delta_i^j\ , \\ &[\hat x^i,\,\hat\xi^\alpha]_- = 0\ , \quad [\hat p_i,\,\hat\xi^\alpha]_- = 0\ , \quad [\hat\xi^\alpha,\,\hat\xi^\beta]_+ = \imath\hbar g^{\alpha\beta}\ . \end{aligned} \tag{36}$$

The appropriate Hilbert space for such QM is $L^2(\mathbb{R}^m,\,\mathrm{d}\mu)\otimes\mathbb{C}^d$, $d=2^{[\frac{n}{2}]}$, and the dynamical equations (in Heisenberg picture) coincide with (30), (31) if we put $\tilde\Omega=0$ and replace $\{.\,,\,.\}_{\tilde\Omega}$ by $[.\,,\,.]_\mp$. Now we are ready to present simple examples.

SUSY-QM: In order to simplify the exposition we put $g^{\alpha\beta}=\delta^{\alpha\beta}$, $m=1$ and $n=2$ (it is well known from the theory of Clifford algebras [21] that for all even n and all metrics $g^{\alpha\beta}$ the properties of the algebras are similar). The supercommutation relations (36) are represented in the Hilbert space $L^2(\mathbb{R},\,\mathrm{d}x)\otimes\mathbb{C}^2$ by the operators

$$\hat x = x\ , \quad \hat p = \imath\hbar\frac{\mathrm{d}}{\mathrm{d}x}\ , \quad \hat\xi^1 = \sqrt{\frac{\hbar}{2}}\begin{pmatrix} 0 & \imath \\ 1 & 0\end{pmatrix}\ , \quad \hat\xi^2 = \sqrt{\frac{\hbar}{2}}\begin{pmatrix} 0 & 1 \\ \imath & 0\end{pmatrix}\ . \tag{37}$$

It is convenient to form from self-adjoint operators $\hat x$ and $\hat p$ the bosonic ladder operators

$$b = \frac{1}{\sqrt{2\hbar}}(\hat x - \imath\hat p)\ , \qquad b^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat x + \imath\hat p)\ , \tag{38}$$

and analogically, from anti-self-adjoint operators $\hat\xi^\alpha = \imath(\hat\xi^\alpha)^\dagger$ $(\alpha=1,2)$ the fermionic ladder operators

$$f = \frac{1}{\sqrt{2\hbar}}(\hat\xi^1 - \imath\hat\xi^2)\ , \qquad f^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat\xi^2 - \imath\hat\xi^1)\ . \tag{39}$$

The general even classical Hamiltonian quadratic in the momentum p is of the form

$$H = \frac{1}{2}(p^2 + V_1^2(x)) + V_2(x)\xi^2\xi^1\ . \tag{40}$$

The most interesting case is $V_2(x) = \pm V_1'(x)$, because then there exist odd conserved quantities ($\{H,\,q\}=0$)

$$q_{1\pm} = p\xi^1 \mp V_1(x)\xi^2\ , \qquad q_{2\pm} = p\xi^2 \mp V_1(x)\xi^1\ , \tag{41}$$

such that

$$\{q_{i+},\,q_{j+}\} = 2\delta_{ij}H_+\ , \qquad \{q_{i-},\,q_{j-}\} = 2\delta_{ij}H_-\ , \tag{42}$$

where

$$H_\pm = \frac{1}{2}(p^2 + V_1^2(x)) \pm V_1'(x)\xi^2\xi^1\ .$$

The corresponding Hamiltonian vector fields ζ_q generate a special kind of internal (exact Cartan) symmetry, which turns into supersymmetry on quantum level. Quantization procedure leads to the Hamiltonian

$$\hat H_\pm = \frac{1}{2}\left(\hat p^2 + V_1^2(\hat x) \pm \hbar V_1'(\hat x)\sigma_3\right)\ , \tag{43}$$

[3] which defines the so called quantum deformation of the classical space of observables

which is the famous Witten's Hamiltonian for the SUSY-QM (proposed in [22],[23] and studied in [24],[25]). The operators $\hat{Q}_{i\pm} = \frac{1}{\sqrt{\imath\hbar}}\hat{q}_{i\pm}$ are odd generators of quantum supersymmetry

$$[\hat{Q}_{i+}, \hat{Q}_{j+}]_+ = 2\delta_{ij}\hat{H}_+ , \qquad [\hat{Q}_{i-}, \hat{Q}_{j-}]_+ = 2\delta_{ij}\hat{H}_- . \tag{44}$$

Very nice and pedagogical articles about SUSY-QM may be found in [26], [27] (see also [28], where SUSY-QM is built from classical mechanics in a Lagrangian framework).
Spin $\frac{1}{2}$ particle: Another important example is $g^{\alpha\beta} = \delta^{\alpha\beta}$, $m = 3$ and $n = 3$. The operator realization of (36) is for even variables the same as in (37), while odd variables are represented by the operators

$$\hat{\xi}^\alpha = \sqrt{\frac{\imath\hbar}{2}}\sigma^\alpha \quad \alpha = 1, 2, 3 , \tag{45}$$

where σ^α are Pauli matrices (for a general odd n the same is true with the σ's replaced by the generators of corresponding Clifford algebra). Classical even Hamiltonian

$$H = \frac{1}{2m}(\vec{p} + q\vec{A})^2 + q\varphi + \tilde{g}\frac{q}{2m}\frac{\partial A_\beta}{\partial x^\alpha}\xi^\alpha\xi^\beta , \tag{46}$$

corresponds after quantization to the well known Pauli Hamiltonian, which describes non-relativistic spin $\frac{1}{2}$ particle with charge q, mass m and Landé $\tilde{g}$-factor in an external electromagnetic field (more details about quantum mechanics related to Pauli equation could be found e. g. in [29]). For the Hamiltonian (46) and a purely magneto-static field described by the vector potential $\vec{A}$ it is possible to find a classical odd conserved quantity

$$Q = \frac{1}{\sqrt{m}}(p_\mu + qA_\mu)\left(\xi^\mu + q(\tilde{g} - 2)\,\frac{\epsilon^{\mu\alpha\beta}\partial_{x^\alpha}A_\beta}{2(\vec{p} + q\vec{A})^2}\,\xi^1\xi^2\xi^3\right) , \tag{47}$$

which obeys $\{Q, Q\} = 2H$. The corresponding Hamiltonian vector field ζ_Q is an exact Cartan symmetry (more details about supersymmetry related to Pauli equation in specific configurations of magnetic field see in [30]).
Quantization(?) of odd supersymplectic structures: The Darboux theorem for odd symplectic structures (for more details and a proof see [31], [32]) states that an arbitrary $m|m$-dimensional odd symplectic supermanifold $\mathfrak{M}$ is locally isomorphic to $\mathbb{R}^{m|m}$ (described by global coordinates $(x^1, \ldots, x^m, \xi^1, \ldots, \xi^m)$) with the odd symplectic structure

$$\Omega = \delta_{i\alpha}\psi^i y^\alpha = \delta_{i\alpha}\mathrm{d}x^i \wedge \mathrm{d}\xi^\alpha . \tag{48}$$

The coordinate Hamiltonian vector fields

$$\zeta_{x^i} = -\delta^{i\alpha}\partial_{\xi^\alpha} , \qquad \zeta_{\xi^\alpha} = \delta^{\alpha i}\partial_{x^i} , \tag{49}$$

define elementary Poisson brackets (one usually refers to the Poisson brackets for odd symplectic structures as Buttin brackets)

$$\{x^i, x^j\} = 0 , \quad \{\xi^a, \xi^\beta\} = 0 , \quad \{\xi^\alpha, x^i\} = \delta^{\alpha i} . \tag{50}$$

The well known and simplest example of an odd symplectic (Poisson) supermanifold is ΠT^*M (the odd cotangent bundle associated to T^*M by changing parity in fibres). Functions on ΠT^*M correspond to multivector fields on the manifold M and Poisson bracket of two functions is defined as Schouten bracket of congruent multivector fields.

It is not fully clear (at least for us) what exactly means the "quantization" of odd symplectic structures. It is possible to adopt the quantization schema from the even case, deforming the algebra $C^\infty(\mathbb{R}^{m|m})$ by introducing the associative $\star$ product

$$f \star g := fg + \{f, \kappa g\} = fg + \kappa\left(\frac{\partial f}{\partial \xi^\alpha}\delta^{\alpha i}\frac{\partial g}{\partial x^i} + (-1)^{\tilde f}\frac{\partial f}{\partial x^i}\delta^{i\alpha}\frac{\partial g}{\partial \xi^\alpha}\right), \tag{51}$$

with odd deforming parameter κ in order to preserve parity. Unfortunately, such associative star product is $\mathbb{Z}_2$-graded commutative and it may be proved (see [33]) that the $\star$ product and the ordinary (exterior) product are equivalent[4]. It was shown by P. Ševera (for more details see [34]) that an odd Poisson structure (in particular, an odd symplectic structure) on arbitrary $m|m$-dimensional smooth supermanifold $\mathfrak{M}$ may be used to deform/quantize the algebra of pseudodifferential forms over $\mathfrak{M}$.

Let us stress, finally, that odd symplectic structure is a crucial ingredient of Batalin-Vilkovisky formalism (for more details see [35],[36]), whose geometrical background is discussed in [32],[37]-[40].

Acknowledgement

I would like to express my deep gratitude to my teacher P. Ševera, who was an excellent guide of mine to supergeometry and who is the originator of many ideas in this article. I am also very grateful to P. Prešnajder and V. Balek who gave me much advice, many thanks go to my girlfriend Tulka (♡) for her mental encouragement.

References

[1] F. A. Berezin, A. A. Kirillov; *Introduction to superanalysis*, Reidel, Dordrecht (1987).

[2] Yu. I. Manin; *Gauge fields theory and complex geometry*, Springer-Verlag, Berlin (1988).

[3] D. A. Leites; Russian Math. Surveys **35**:1 (1980) 1.

[4] B. S. DeWitt; *Supermanifolds*, University Press, Cambridge (1984).

[5] V.S. Vladimirov, I.V. Volovich; Teor. Mat. Fiz. **59** 3 (1984) 317.

[6] V.S. Vladimirov, I.V. Volovich; Teor. Mat. Fiz. **60** 2 (1984) 743.

[7] M.Scheunert; *The theory of Lie superalgebras*, Springer-Verlag, Berlin (1979).

[8] P. Nelson; *Introduction to Supermanifolds*, HUTP-86/A024.

[9] M. Kontsevich; *Deformation quantization of Poisson manifolds, I*, q-alg/9709040.

[10] A. Frölicher, A. Nijenhuis; Indag. Math. **29** (1956) 338.

[11] V. I. Arnol'd; *Mathematical methods of classical mechanics*, Springer-Verlag, New York, NY (1978).

[12] R. Abraham, J. E. Marsden; *Foundations of mechanics*, Benjamin, New York, NY (1978).

[13] J. L. Martin; Proc. Roy. Soc. **A 251** (1959) 536.

[14] F. A.Berezin, M. S. Marinov; JETP Lett **21** (1975) 320.

[4] another possible location of κ leads to a different supercommutative product or to a $\star$ product that is not associative

[15] F. A.Berezin, M. S. Marinov; Ann. Phys. **104** (1977) 336.

[16] R. Casalbuoni; Nuovo Cimento **A 33** (1976) 115 and 389.

[17] R. Casalbuoni; Phys. Lett **B 62** (1976) 49.

[18] B. Fedosov; *Deformation quantization and index theory*, Akademie Verlag, Berlin (1995).

[19] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer; Ann. Phys. NY **111** (1978) 61 and 111.

[20] A. C. da Silva, *Lectures on symplectic geometry*, Springer-Verlag, Berlin (2001).

[21] J. Snygg; *Clifford algebra: a computational tool for physicists*, Oxford University Press, New York, NY (1997).

[22] E. Witten; Nucl. Phys. **B 185** (1981) 513.

[23] E. Witten; Nucl. Phys. **B 202** (1982) 253.

[24] P. Salomonson, J. W. van Holten; Nucl. Phys. **B 196** (1982) 509.

[25] M. de Crombrugghe, V. Rittenberg; Ann. Phys. NY **151** (1983) 99.

[26] L. E. Gendenshtain, I. V. Krive; Sov. Phys. Usp. **28** (1985) 645.

[27] H. Grosse; *Supersymmetric Quantum Mechanics*, UWThPh-1989-54, Lectures given at the Brasov International School on "Recent Developments in Quantum mechanics" (1989).

[28] F. Ravndal; *Elementary supersymmetry*, CERN 85-11, CERN School of Physics (Proceedings) (1984) 300.

[29] L. D. Landau, E. M. Lifshitz; *Quantum mechanics - non-relativistic theory* 3rd ed., Pergamon Press, Oxford (1977).

[30] J. Niederle, A. G. Nikitin; J. Math. Phys. **40** (1999) 1280.

[31] V. N. Shander; Comptes rendus de l' Academic bulgare des Sciences **36** n.3 (1983) 309.

[32] H. O. M. Khudaverdian; *Semidensities on Odd Symplectic Supermanifold*, math.DG/0012256.

[33] D. Tamarkin, B. Tsygan; *Noncommutative differential calculus, homotopy BV algebras and formality conjectures*, math.KT/0002116.

[34] P. Ševera; *Quantization of the odd symplectic category*, math.QA/0210169.

[35] I. A. Batalin, G. A. Vilkovisky; Phys. Lett. **102 B** (1981) 27.

[36] I. A. Batalin, G. A. Vilkovisky; Nucl. Phys. **B 234** (1984) 106.

[37] H. O. M. Khudaverdian; J. Math. Phys. **32** (1991) 1934.

[38] H. O. M. Khudaverdian, A. P. Nersessian; Mod. Phys. Lett. **A 8** No. 25 (1993) 2377.

[39] H. O. M. Khudaverdian, A. P. Nersessian; J. Math. Phys. **37** (1996) 3713.

[40] A. S. Schwartz; Commun. Math. Phys. **155** (1993) 249.